ALMOST SURE CONVERGENCE AND BOUNDED ENTROPY

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ABSTRACT

It is shown that the almost sure convergence property for certain sequences of operators $\{S_n\}$ implies a uniform bound on the metrical entropy of the sets $\{S_n f \mid n = 1, 2, \dots\}$, where f is taken in the L^2-unit ball. This criterion permits one to unify certain counterexamples due to W. Rudin [Ru] and J. M. Marstrand [Mar] and has further applications. The theory of Gaussian processes is crucial in our approach.

1. Introduction

Let f be a 1-periodic measurable function on $\mathbf{R}$ (which can thus be seen as a function on the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$). For $n = 1, 2, \dots$ define as follows the Riemann sum operators:

$$R_n f(x) = \frac{1}{n} \sum_{0 \leq j < n} f\left(x + \frac{j}{n}\right). \tag{1}$$

For $f \in L^1(\mathbf{T})$, it is clear that $R_n f$ converges to $\int_0^1 f(x)dx$ in the mean. On the other hand, it was shown by W. Rudin [Ru] that there is not necessarily almost sure convergence, even for bounded functions, solving a problem of Marcinkiewicz and Zygmund [M–Z]. This result complements the theorem of Jessen [J] according to which $(R_{n_k} f)$ is almost surely convergent for $f \in L^1(\mathbf{T})$, provided (n_k) is a sequence of integers satisfying $n_k \mid n_{k+1}$.

The next result I am recalling is Marstrand's [Mar] (negative) solution of Khintchine's conjecture [Kh]. Marstrand produced measurable subsets A of $\mathbf{T}$ for which the averages

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$$\frac{1}{n}\sum_{1\le j\le n} f(jx), \qquad f=\chi_A \tag{2}$$

do not converge almost surely.

The purpose of this paper is to state a necessary condition for almost sure convergence related to certain sequences of operators $(S_n)_{n=1,2,\ldots}$. This condition has a variety of applications, to problems of harmonic analysis and ergodic theory, including the Riemann sum problem and the Khintchine problem. Roughly speaking, the condition is a uniform bound on the metrical entropy of the sets $\{S_n f \mid n=1,2,\ldots\}$ as subsets of L^2, when f ranges in the L^2-unit ball. The more precise statement will appear in the next section. It turns out that this condition is violated as well for the operators $S_n = R_n$ as for the averaging operators given by (2). This leads to a certain unification of both counterexamples. As further application of our method, a problem due to A. Bellow [Be] and a question raised by P. Erdös [Er] are settled.

In [Be], the following question is considered. Is it true that for any sequence (a_n) of real numbers converging to 0, there are L^1-functions such that the averages

$$\frac{1}{n}\sum_{j\le n} f(x+a_j) \tag{3}$$

are not almost surely converging (to f)? The answer is affirmative and one may even produce f to be bounded (or the indicator function of a set).

The problem of Erdös mentioned above deals with weaker versions of the Khintchine problem. In particular he raised the question whether given a measurable subset A of $\mathbf{T}$, then for almost all x the set $\{j\in\mathbf{Z}_+ \mid jx\in A\}$ has a logarithmic density, i.e.

$$\frac{1}{\log n}\sum_{\substack{j\le n\\ jx\in A}}\frac{1}{j}\to |A|.$$

We will disprove this fact.

The hypothesis on the operators S_n, when stating the entropy condition, is the commutation with another sequence of operators (positive isometries) satisfying the mean ergodic thoerem. This condition is more general than assuming the S_n commute with group translations (cf. [St]). In the context of operators S_n commuting with group translations, our results are of course closely related (and complement in some sense) E. Stein's paper [St] mentioned above. The reason for adopting a more general point of view is to cover, for instance, the operators given by (3). The description of these operators in terms of group actions does not seem natural.

2. The entropy criterion

The more general context we adopt here originates from the applications we intend to discuss. However, hypotheses and statements made do not necessarily have the most general formulation.

Our method is probabilistic and uses the theory of Gaussian processes. To the Gaussian process $X_t(\omega)$ indexed by $t \in T$, associate the distance on T

$$d(t, t') = \| X_t - X_{t'} \|_2.$$

Obviously T, d as a metric space embeds in Hilbert space. We will be interested in the quantity

$$\int \sup_{t \in T} |X_t(\omega)| \, d\omega \tag{4}$$

which, by Slépian's lemma, is determined by the geometry (in the metrical sense) of T, d. According to [Du], call a subset A of a Hilbert space H a GB-set provided (4) is bounded whenever T, d embeds in A (in the Lipschitz sense). By Sudakov's inequality, such a set satisfies in particular the entropy estimate

$$\sup_{\delta > 0} \delta [\log N(A, \delta)]^{1/2} < \infty \tag{5}$$

where $N(A, \delta)$ refers to the δ-entropy numbers of A, i.e. the numbers of δ-balls needed for covering.

Let (X, μ) be a probability space and (S_n) a sequence of norm $\leqq 1$ operators on $L^2(\mu)$. Assume the existence of a sequence $(T_j)_{j=1,2,\dots}$ of positive (into) isometries, satisfying $T_j(1) = 1$ and the following conditions:

(6) The T_j satisfy a mean ergodic theorem, i.e.

$$\frac{1}{J} \sum_{j \leq J} T_j f \to \int f \text{ in the mean}, \qquad \text{for } f \in L^1.$$

(7) The sequences (T_j) and (S_n) are commuting, i.e. $T_j S_n = S_n T_j$.

Proposition 1. *Assume $p < \infty$ and $(S_n f)$ converges a.s. (almost surely)* *for all* $f \in L^p(\mu)$. *Then, for each* $f \in L^2$, *the set* $\{S_n f \mid n = 1, 2, \ldots\}$ *is a GB-subset of* L^2. *In particular, there is a uniform entropy estimate*

$$\delta(\log N_f(\delta))^{1/2} < C, \qquad \delta > 0, \quad \| f \|_2 \leqq 1$$

where $N_f(\delta)$ *refers to the* δ*-entropy number of the set* $\{S_n f \mid n = 1, 2, \ldots\}$.

PROPOSITION 2. *Assume that* $(S_n f)$ *is a.s. convergent for bounded measurable functions* f. *Then, with the notation of Proposition* 1, *there are uniform entropy estimates*

$$N_f(\delta) \leqq C(\delta), \qquad \| f \|_2 \leqq 1$$

for $\delta > 0$.

The applications presented in the next section use Proposition 2, since it is shown that $N_f(\delta)$ is unbounded for some constant $\delta > 0$.

Observe that the operators T_j considered above satisfy

$$T_j(f^2) = T_j(f)^2, \qquad f \in L^\infty$$

(this follows from the fact that disjoint indicator-functions are mapped on disjoint indicator-functions).

The hypothesis of both Proposition 1 and Proposition 2 has an implication on the behavior of the maximal operator $\sup_n |S_n f|$.

Under the hypothesis of Proposition 1, there is a function $C(\varepsilon)$, $\varepsilon > 0$, such that

$$\mu\left[\sup_n |S_n f| \leqq C(\varepsilon)\right] > 1 - \varepsilon \qquad \text{whenever } f \in L^\infty, \quad \| f \|_p \leqq 1 \tag{8}$$

(we assume here $p \geqq 2$).

Under the hypothesis of Proposition 2, there is a function $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$ such that

$$\int \sup |S_n f| \, d\mu < \delta(\varepsilon), \qquad \text{whenever } |f| \leqq 1, \quad \int |f| \, d\mu < \varepsilon. \tag{9}$$

The verification of these two statements are left to the reader.

PROOF OF PROPOSITION 1. Fix an integer N and denote $\bar{N}$ the set $\{1, \ldots, N\}$. Letting $\{g_j\}_{j=1,\ldots,J}$ be a sequence of independent Gaussian random variables, define for $f \in L^\infty$, $\| f \|_2 \leqq 1$

$$F_\omega(x) = J^{-1/2} \sum_{j \leqq J} g_j(\omega) T_j f. \tag{10}$$

Here J will be taken large enough depending on f and an N. By the commutation property (7),

$$S_n F_\omega = J^{-1/2} \sum_{j \leqq J} g_j(\omega) T_j S_n f. \tag{11}$$

By (8), we have for each ω

$$\mu\left[\sup_{n \leqq N} |S_n F_\omega| \leqq C(\varepsilon) \| F_\omega \|_p\right] > 1 - \varepsilon. \tag{12}$$

One clearly has

$$\int \| F_\omega \|_p d\omega \leqq \| F_\omega(x) \|_{L^p(d\omega \otimes dx)} \leqq \sqrt{p} \left\| \left(\frac{1}{J} \sum_{j \leqq J} | T_j f |^{1/2}\right)^{1/2} \right\|_p \tag{13}$$

where

$$\left\| \frac{1}{J} \sum_{j \leqq J} | T_j f |^2 \right\|_{p/2} = \left\| \frac{1}{J} \sum_{j \leqq J} T_j(f^2) \right\|_{p/2} \leqq 2 \| f \|_2^2 \tag{14}$$

by hypothesis (6), letting J be large enough (the boundedness of f is used here). Hence, by (13), (14)

$$\int \| F_\omega \|_p d\omega \leqq 2. \tag{15}$$

Denote for convenience $F_\omega^* = \sup_{n \leqq N} | S_n F_\omega |$. Since for each x there is uniform equivalence

$$\| F_\omega^*(x) \|_{L^1(d\omega)} \sim \| F_\omega^*(x) \|_{L^2(d\omega)};$$

clearly, for some constant $c > 0$

$$(\mu \otimes \mathbf{P})\left[(x, \omega) \,\middle|\, F_\omega^*(x) > c \int F_\omega^*(x) d\omega\right] > c. \tag{16}$$

In the sequel, (different) constants will be denoted by letters $c > 0$, $C < \infty$. From (15), (16) follows the existence of some $\bar{\omega}$ satisfying

$$\begin{gathered} \| F_{\bar\omega} \|_p \leqq C, \\ \mu\left[F_{\bar{\omega}}^*(x) > c \int F_\omega^*(x) d\omega\right] > c. \end{gathered} \tag{17}$$

By (12)

$$\mu[F_{\bar{\omega}}^* \leqq C C(\varepsilon)] > 1 - \varepsilon. \tag{18}$$

Choosing $\varepsilon > 0$ appropriately in (18), (17) and (18) yield a subset $X' \subset X$, $\mu(X') > \varepsilon$ and for $x \in X'$

$$\int F_\omega^*(x) d\omega < C.$$

Hence, by (11)

$$\int \sup_{n \leq N} \left| J^{-1/2} \sum_{j \leq J} g_j(\omega)(T_j S_n f)(x) \right| d\omega < C. \tag{19}$$

for $x \in X$, consider the distance on $\bar{N}$

$$d_x(n, n') = \left[\frac{1}{J} \sum_{j \leq J} | T_j(S_n f)(x) - T_j(S_{n'} f)(x) |^2 \right]^{1/2}$$

$$= \left[\frac{1}{J} \sum_{j \leq J} T_j | S_n f - S_{n'} f |^2(x) \right]^{1/2}.$$

Again by (6), a sufficiently large value of J will permit one to ensure that

$$d_x(n, n') \approx \| S_n f - S_{n'} f \|_2, \qquad n, n' \in \bar{N} \tag{20}$$

for x in a set of almost full measure and hence for some $x \in X'$, for which (19) holds. Since the constant C in (19) does not depend on N, (19) and (20) imply that $\{S_n f\}$ is a GB-set.

The second claim in Proposition 1 is Sudakov's minoration.

PROOF OF PROPOSITION 2. Assume $\delta > 0$ such that $N_f(\delta)$ is unbounded over the L^2-unit ball. Fix K and let $f \in L^\infty$, $\| f \|_2 = 1$ be such that for some $I \subset \mathbf{Z}_+$, $\| I \| = K$

$$\| S_n f - S_{n'} f \|_2 > \delta \qquad \text{for } n \neq n' \text{ in } I. \tag{21}$$

Let F_ω be again defined by (10). Write

$$F_\omega = \varphi_\omega + G_\omega$$

where

$$\varphi_\omega = F_\omega \chi_{[|F_\omega| < \beta\sqrt{\log K}]} \quad \text{and} \quad G_\omega = F_\omega \chi_{[|F_\omega| > \beta\sqrt{\log K}]}$$

and β is a constant to be specified later.

Since the S_n are L^2-contractions.

$$\| S_n G_\omega \|_2 \leq \| G_\omega \|_2. \tag{22}$$

For $\lambda > 0$, one has

$$(23)\qquad \int\int \exp[\lambda F_\omega(x)]d\omega dx \leqq \int \exp\left[\frac{\lambda^2}{J}\sum_{j\leqq J} T_j(f^2)\right]d\mu \leqq \exp(2\lambda^2)$$

again by (6), for sufficiently large J. Define

$$\mu_t(\omega)=\mu[x\in X \mid |F_\omega(x)|>t].$$

It follows from (23) that

$$e^{\lambda t}\int \mu_t(\omega)d\omega \leqq e^{2\lambda^2}$$

hence for appropriate λ

$$(24)\qquad \int \mu_t(\omega)d\omega \leqq e^{-t^2/9}.$$

By choosing J sufficiently large, (24) will be valid for t in an arbitrarily chosen finite interval $[0, T]$. If f satisfies $|f| \leqq B$ pointwise, for some B, $\int |F_\omega(x)|^2 d\omega \leqq B^2$ pointwise and hence clearly

$$(25)\qquad \int \mu_t(\omega)d\omega \leqq \exp\left(-\frac{1}{2}\frac{t^2}{B^2}\right).$$

Estimate by (24), (25)

$$\int\left[\int |G_\omega|^2 dx\right]d\omega \leqq 2\int\left\{\int_{\beta\sqrt{\log K}}^{\infty} t\mu_t(\omega)dt\right\}d\omega$$

$$\leqq 2\int_{\beta\sqrt{\log K}}^{T} t\exp\left(-\frac{t^2}{9}\right)dt + 2\int_T^\infty t\exp\left(-\frac{1}{2}\frac{t^2}{B^2}\right)dt$$

hence

$$(26)\qquad \int \| G_\omega \|_2^2\, d\omega < \frac{1}{K}$$

for appropriate choice of β and T.

Consequently, by (22), (26)

$$(27)\qquad \begin{aligned}\int\left[\int \sup_{n\in I} |S_n(G_\omega)|dx\right]d\omega &\leqq \int\left(\sum_I \| S_n(G_\omega)\|_2^2\right)^{1/2} d\omega\\ &\leqq K^{1/2}\int \| G_\omega \|\, d\omega < 1.\end{aligned}$$

Denote again $F_\omega^* = \sup_{n\in I}|S_n(F_\omega)|$. From (16) follows the existence of a set A,

$\mathbf{P}(A) > c$ such that to each $\omega \in A$ corresponds a set $X_\omega \subset X$, $\mu(X_\omega) > c$ on which

$$F_\omega^*(x) > c \int F_{\omega'}^*(x) d\omega' \qquad (x \in X_\omega). \tag{28}$$

Hence, for $\omega \in A$, by Sudakov's inequality

$$\int F_\omega^*(x)dx \geqq \int_{X_\omega} F_\omega^*(x)dx$$

$$> c\delta(\log K)^{1/2} \mu[x \in X_\omega \mid d_x(n, n') > \delta/2 \text{ for } n \neq n' \text{ in } I].$$

Thus, for J sufficiently large, by (20), (21)

$$\int F_\omega^*(x)dx \geqq c\delta(\log K)^{1/2} \qquad \text{for } \omega \in A. \tag{29}$$

Writing

$$\sup_{n \in I} |S_n(\varphi_\omega)| \geqq F_\omega^* - \sup_I |S_n(G_\omega)|$$

(27), (29) yield a subset A' of A, $\mathbf{P}(A') > c$ on which

$$\int \sup_I |S_n(\varphi_\omega)| d\mu > c\delta(\log K)^{1/2} \qquad (\omega \in A'). \tag{30}$$

Also

$$\int \| \varphi_\omega \|_1 d\omega \leqq \int \| F_\omega \|_1 d\omega \leqq 1. \tag{31}$$

From (30), (31), a point $\bar{\omega}$ is obtained fulfilling the properties

$$\| \varphi_{\bar\omega} \|_1 < C,$$

$$\left\| \sup_I |S_n(\varphi_{\bar\omega})| \right\|_1 > c\delta(\log K)^{1/2}.$$

Define $\psi = (\log K)^{-1/2} \varphi_{\bar\omega}$ satisfying $|\psi| \leqq \beta$ and $\int \sup |S_n \psi| d\mu > c\delta$. Moreover

$$\int |\psi| d\mu < c(\log K)^{-1/2} \to 0 \qquad \text{as } K \to \infty.$$

This contradicts (9) and completes the proof.

3. Applications

A first application is the case of "invariant operators" in the setting of [St]. Let G be a compact metrizable group with invariant measure ν. Let Ω be an homogeneous space of G and μ its G-invariant measure. Denote τ_g the translation operator

$$\tau_g f(x) = f(g^{-1}(x)) \qquad \text{for } g \in G.$$

For random sequences $\{g_j\}$ in G, the sequence $T_j = \tau_{g_j}$ will almost surely satisfy condition (6). Hence

PROPOSITION 3. *Let (S_n) be a sequence of uniformly bounded operators on $L^2(\Omega)$ commuting with the translations $\tau_g, g \in G$. Then the statements in Propositions 1 and 2 are valid.*

In the context of the Marcinkiewicz–Zygmund problem for the Riemann sums R_n defined by (1), G is the circle group $\mathbf{T}$ and $X = G$, with trivial action.

PROPOSITION 4 ([Ru]). *There are bounded measurable functions f on $\mathbf{T}$ for which $R_n f$ does not converge a.s.*

PROOF. By Proposition 2, it suffices to show that for some $\delta > 0$ the entropy-numbers $N_f(\delta)$ are not uniformly bounded for $\| f \|_2 \leqq 1$. Notice that as a Fourier multiplier R_n acts the following way:

$$R_n f(k) = \hat{f}(k) \quad \text{if } k \text{ is a multiple of } n,$$

$$= 0 \qquad \text{otherwise.}$$

Choose a sequence $p_1, p_2, \ldots, p_r$ of distinct prime numbers. Denote E the set of 2^r simple products of $p_1, \ldots, p_r$ and consider the function

$$f = 2^{-r/2} \sum_{n \in E} e^{2\pi i n x}$$

Then

$$R_{p_s} f = 2^{-r/2} \sum_{n \in (E \cap p_s \mathbf{Z})} e^{2\pi i n x}$$

and thus, for $1 \leqq s \neq t \leqq r$,

$$\| R_{p_s} f - R_{p_t} f \|_2 = 2^{-r/2} | (E \cap p_s \mathbf{Z}) \Delta (E \cap p_t \mathbf{Z}) | = 2^{-r/2} (2^{r-1})^{1/2} = 1/\sqrt{2}.$$

Consequently, for $\delta = 1/\sqrt{2}$, $\sup_{\| f \|_2 \leq 1} N_f(\delta) = \infty$.

The next corollary is of relevance for the Khintchine problem.

PROPOSITION 5. *Let (T_j) be a sequence of positive commuting isometries, $T_j(1)=1$, satisying condition (6). Then Proposition 1 and Proposition 2 apply to any sequence of operators S_n obtained as convex combinations of the T_j.*

Define $T_j f(x) = f(jx)$. Applying Proposition 2, the existence of bounded measurable functions f for which

$$S_n f = \frac{1}{n} \sum_{j \leqq n} T_j f$$

does not converge almost surely, is a consequence of

LEMMA 6. *There is $\delta > 0$ such that $N_f(\delta)$ is not uniformly bounded for $f \in L^2(\mathbf{T})$, $\|f\|_2 \leqq 1$.*

PROOF. The construction described below is part of Marstrand's approach [Mar]. Denote $p_1, p_2, \ldots$ the sequence of consecutive prime numbers. If $n = p_1^{\alpha_1} p_2^{\alpha_2}, \ldots$, we let n correspond to $(\alpha_1, \alpha_2, \ldots)$ and will replace the multiplicative problem by an additive problem.

Fix an integer s and denote for each T

$$A_T = \{(\alpha_1, \ldots, \alpha_s) \in \mathbf{Z}_+^s \mid T \leqq \alpha_1 \log p_1 + \cdots + \alpha_s \log p_s < T+1\}$$

where $\mathbf{Z}_+$ denotes the positive integers including 0 and the logarithm is taken in basis 2. Thus A_T corresponds to the set of integers $2^T \leqq n < 2.2^T$ which prime divisors are contained in the set $p_1, \ldots, p_s$.

Since $p_1 = 2$, replacement of α_1 by $\alpha_1 + 1$ shows that

$$|A_{T+1}| \geqq |A_T| \tag{32}$$

while obviously

$$|A_T| \leqq T^s \tag{33}$$

which (for s fixed) is a polynomial growth. Thus (32) and (33) yield some T such that the sets $A_T, A_{T+1}, \ldots, A_{T+d}$ are of comparable size, i.e.

$$B < |A_{T+i}| < 2B \qquad (0 \leqq i \leqq d). \tag{34}$$

Here d is any preliminary chosen integer and T depends on d. We let $2^d < s$. Defining

$$V_i = \{n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \mid (\alpha_1, \ldots, \alpha_s) \in A_{T+i}\},$$

$$f^{(i)} = B^{-1/2} \sum_{n \in V_i} e^{2\pi i n x},$$

clearly $1 \leqq \| f^{(i)} \|_2 \leqq 2$ and $f^{(i)} \perp f^{(i')}$ for $0 \leqq i \neq i' \leqq d$ since the sets V_i are disjoint.

Assume $n \in V_0$ and $2^{i-1} \leqq j \leqq 2^i < s$. By construction nj only contains prime factors in $p_1, \ldots, p_s$ and satisfies

$$T + i - 1 < T + \log j \leqq \log nj < T + 1 + \log j \leqq T + i + 1.$$

Hence

$$jV_0 \subset V_i \cup V_{i-1}$$

and thus, for $f = f^{(0)}$,

$$\langle T_j f, f^{(i-1)} + f^{(i)} \rangle \geqq B^{-1} |jV_0| \geqq 1 \tag{35}$$

by (34). Since (35) holds whenever $2^{i-1} \leqq j \leqq 2^i$, also

$$\langle S_{2^i} f, f^{(i-1)} + f^{(i)} \rangle \geqq \tfrac{1}{2}. \tag{36}$$

Defining $\varphi_i = 2^{-1/2}[f^{(2i-1)} + f^{(2i)}]$ for $1 \leqq i < \bar{i} = [d/2]$, $(\varphi_i)_{i \leqq \bar{i}}$ is an orthonormal sequence and $\langle S_{4^i} f, \varphi_i \rangle > \frac{1}{4}$. It is now an elementary verification to see that the $\frac{1}{10}$-entropy-number of $\{S_{4^i} f \mid i < \bar{i}\}$ as a subset of $L^2(\mathbf{T})$ is at least cd. Hence, we have proved that $N_f(\frac{1}{10})$ is not uniformly bounded for $\| f \|_2 \leqq 1$.

Remark. Koksma [Ko] has given a sufficient condition on the Fourier coefficients of f in order to ensure a.s. convergence of $(1/n) \sum_{j \leqq n} f(jx)$. A more detailed analysis of previous construction shows that his double logarithmic condition is essentially best possible.

P. Erdös considered weaker versions of the Khintchine problem (Louisiana State University, November 1987) and, in particular, the question whether given a measurable subset A of $\mathbf{T}$ it is true that for almost all x the set $\{j \in \mathbf{Z}_+ \mid jx \in A\}$ has a logarithmic density, i.e.

$$\frac{1}{\log n} \sum_{\substack{j \leqq n \\ jx \in A}} \frac{1}{j} \to |A|.$$

This fact may be disproved using the same method as above. More generally

Proposition 7. *Let $(\lambda_j)_{j=1,2,\ldots}$ be a decreasing sequence of positive numbers such that $\sum \lambda_j = \infty$ and define*

$$S_n f(x) = \frac{1}{\sigma_n} \sum_{j \le n} \lambda_j f(jx), \qquad \sigma_n = \sum_{j \le n} \lambda_j .$$

Then there is a bounded measurable function for which $S_n f$ does not converge a.s. disproving the existence of a.s. convergent summation procedures.

As in the Khintchine problem, it will be shown that $N_f(\delta)$ is unbounded for some fixed $\delta > 0$. A modification of the previous construction seems needed.

LEMMA 8. *Let $p_1, \dots, p_s$ be a sequence of prime numbers, $\sum p_i^{-1} > 1$. Denote for $j \in \mathbf{Z}_+$ by $\beta(j) = \alpha_1 + \cdots + \alpha_s$ where $\alpha_1, \dots, \alpha_s$ are the respective exponents of $p_1, \dots, p_s$ in the prime decomposition of j. Let $N > (\sum p_i)^2$ and I be the interval $[0, N] \cap \mathbf{Z}$. Then in the following deviation estimate*

$$\left| \left\{ j \in I \;\middle|\; \left| \beta(j) - \sum \frac{1}{p_i} \right| > \gamma \right\} \right| \le C \gamma^{-2} \left(\sum \frac{1}{p_i} \right) N. \tag{37}$$

PROOF. For $i = 1, \dots, s$ and $r = 1, 2, \dots$ define as follows the functions $\chi_{i,r}$ on I:

$$\begin{cases} \chi_{i,r}(j) = 1 & \text{if } p_i^r \mid j, \\ \qquad\;\; = 0 & \text{otherwise.} \end{cases}$$

Thus

$$\beta(j) = \sum_{i=1}^{s} \sum_{r=1}^{\infty} \chi_{i,r}(j).$$

Clearly

$$p_i^{-r} - \frac{p_i^r}{N} \le N^{-1} \sum_{j \le N} \chi_{i,r}(j) \le p_i^{-r}, \tag{38}$$

$$\left| N^{-1} \sum_{j \le N} \chi_{i,1}(j) \chi_{i',1}(j) - \frac{1}{p_i p_{i'}} \right| \le \frac{p_i p_{i'}}{N}. \tag{39}$$

Estimate

$$\begin{aligned} \left(N^{-1} \sum_{j \le N} \left| \beta(j) - \sum_{i=1}^{s} \frac{1}{p_i} \right|^2 \right)^{1/2} &\le \left(N^{-1} \sum_{j \le N} \left| \sum_i \chi_{i,1}(j) - \sum \frac{1}{p_i} \right|^2 \right)^{1/2} \\ &\quad + \sum_{r \ge 2} \sum_{i=1}^{s} \left(N^{-1} \sum_{j \le N} \chi_{i,r}(j) \right)^{1/2} \end{aligned} \tag{40}$$

where

$$N^{-1}\sum_{j\leqq N}\left|\sum_i \chi_{i,1}(j)-\sum p_i^{-1}\right|^2=\sum\left(N^{-1}\sum_{j\leqq N}|\chi_{i,1}(j)-p_i^{-1}|^2\right)$$

$$(41)\qquad +2\sum_{i\neq i'}\left\{N^{-1}\sum_{j\leqq N}(\chi_{i,1}(j)-p_i^{-1})(\chi_{i',1}(j)-p_{i'}^{-1})\right\}$$

which, by (38) and (39), is bounded by

$$(42)\qquad 2\sum\frac{1}{p_i}+2\sum_{i\neq i'}\left\{\frac{p_i p_i'}{N}+\frac{p_i'}{N}p_i^{-1}+\frac{p_i}{N}p_{i'}^{-1}\right\}<2\sum\frac{1}{p_i}+CN^{-1}\left(\sum p_i\right)^2$$

and, from (38),

$$(43)\qquad \sum_{r\geqq 2}\sum_{i=1}^{s}\left(N^{-1}\sum_{j\leqq N}\chi_{i,r}(j)\right)\leqq\sum_{i=1}^{s}\left(\sum_{r\geqq 2}p_i^{-r}\right)\leqq C.$$

Collecting estimates (40), (41), (42) and (43), it follows that

$$(44)\qquad \left(N^{-1}\sum_{j\leqq N}\left|\beta(j)-\sum\frac{1}{p_i}\right|^2\right)^{1/2}\leqq 2\left(\sum\frac{1}{p_i}\right)^{1/2}+C<C\left(\sum\frac{1}{p_i}\right)^{1/2}$$

from the hypothesis in Lemma 8. Inequality (37) is now immediate.

LEMMA 9. *Let $p_1,\dots,p_s$ be as in Lemma 8 and let (λ_j) be a decreasing sequence of positive numbers $\leqq 1$ such that, for a given $K>1$,*

$$(45)\qquad \infty>\sum\lambda_j>K^2\left(\sum p_i\right)^2.$$

Then

$$(46)\qquad \frac{1}{\sum\lambda_j}\sum\left\{\lambda_j \;\middle|\; \left|\beta(j)-\sum\frac{1}{p_i}\right|>K\left(\sum\frac{1}{p_i}\right)^{1/2}\right\}<CK^{-2}.$$

PROOF. Estimate by partial summation and Lemma 8

$$\sum\left\{\lambda_j \;\middle|\; \left|\beta(j)-\sum\frac{1}{p_i}\right|>\gamma\right\}\leqq\sum_j(\lambda_j-\lambda_{j+1})\left|\left\{k\leqq j \;\middle|\; \left|\beta(k)-\sum\frac{1}{p_i}\right|>\gamma\right\}\right|$$

$$(47)\qquad \leqq c\sum_j(\lambda_j-\lambda_{j+1})\left\{\gamma^{-2}\left(\sum\frac{1}{p_i}\right)j+\left(\sum p_i\right)^2\right\}$$

$$\leqq c\gamma^{-2}\left(\sum\frac{1}{p_i}\right)\left(\sum\lambda_j\right)+c\left(\sum p_i\right)^2.$$

Then

$$\gamma = K\left(\sum \frac{1}{p_i}\right)^{1/2}.$$

Clearly (47), (45) imply (46).

PROOF OF PROPOSITION 7. We need obviously assume that $\sigma_n \to \infty$. Let $p_1, p_2, \ldots$ be the consecutive prime numbers.

Fix an integer n. Construct by induction sequences of integers $(I_s)_{1 \leqq s \leqq r}$, $(J_s)_{1 \leqq s < r}$ satisfying (M to be specified later)

$$I_s < J_s < I_{s+1},$$
$$\sigma_{J_s} > M^4\left(\sum_{i \leqq I_s} p_i\right)^2, \tag{48}$$

$$P_{s+1} > J_s \qquad \text{where } P_s = \sum_{I_{s-1} \leqq i < I_s} p_i^{-1}. \tag{49}$$

For an integer n, denote

$$\beta_s(n) = \sum_{I_{s-1} \leqq i < I_s} \alpha_i$$

where α_i is the exponent of p_i in the prime decomposition of n.

Thus for $t \leqq s$, by Lemma 9 and (48)

$$\frac{1}{\sum_{j \leqq J_s} \lambda_j} \sum_{j \leqq J_s} \{\lambda_j \mid |\beta_t(j) - P_t| > MP_t^{1/2}\} < CM^{-2}.$$

Notice also that by (48), (49), $P_t > J_{t-1} \geqq \sigma_{J_{t-1}} > M^4$ and hence

$$\sum_{j \in E_s} \lambda_j \leqq CM^{-2} s\left(\sum_{j < J_s} \lambda_j\right) < \frac{1}{2} \sum_{j \leqq J_s} \lambda_j \tag{50}$$

provided $M > r$ and defining

$$E_s = \{j \leqq J_s \mid |\beta_t(j) - P_t| > M^{-1} P_t \text{ for some } t \leqq s\}.$$

Define for system $a_1, \ldots, a_r$ of integers

$$V(a_1, \ldots, a_r) = \{n \mid a_s \leqq \beta_s(n) < a_s + 1 \ (1 \leqq s \leqq r) \text{ and } \beta_s(n) = 0 \text{ for } s > r\}$$

which may be identified with the product $A_1(a_1) x \cdots A_r(a_r)$ where

$$A_s(a) = \left\{(\alpha_i)_{I_{s-1} \leqq i < I_s} \mid a \leqq \sum \alpha_i < a + 1\right\}.$$

The same growth argument as in Lemma 6 then permits one to find integers T_s such that

$$(51)\qquad \begin{aligned} |V(T_1,\dots,T_r)| &\leqq |V(T_1+q_1,\dots,T_r+q_r)| \\ &\leqq 2|V(T_1,\dots,T_r)| \quad \text{if } |q_s|\leqq 2P_s. \end{aligned}$$

Define

$$f=|V|^{-1/2}\sum_{n\in V}e^{2\pi inx}\qquad \text{where } V=\bigcup_{0\leqq q_s\leqq P_s/10}V(T_1+q_1,\dots,T_r+q_r).$$

Next for $s\leqq r$, we analyze $\Sigma_{j\leqq J_s}\lambda_j f(jx)$. Assume $j\notin E_s$ and $n\in V$, $t\leqq s$. Then, since $\beta_t(jn)=\beta_t(j)+\beta_t(n)$.

$$(52)\qquad jV\subset V_s$$

where

$$\begin{aligned} V_s=\bigcup V(T_1+P_1+q_1',\dots,T_s+P_s+q_s',\\ T_{s+1}+q_{s+1}',T_{s+2}+q_{s+2}',\dots,T_r+q_r') \end{aligned}$$

and the union extends over

$$\begin{aligned} &-M^{-1}P_t\leqq q_t'\leqq \tfrac{1}{10}P_t+M^{-1}P_t \qquad (t\leqq s),\\ &0\leqq q_{s+1}'\leqq \tfrac{1}{10}P_{s+1}+\log J_s,\\ &0\leqq q_t'\leqq \tfrac{1}{10}P_t \qquad \text{for } t>s+1. \end{aligned}$$

Notice that from the given estimates (51) and, since $\log J_s<P_{s+1}$ by (49),

$$\frac{|V_s|}{|V|}\leqq C\frac{s}{M}+C\frac{\log J_s}{P_{s+1}}<2.$$

Thus the function

$$g_s=|V|^{-1/2}\sum_{n\in V_s}e^{2\pi inx}$$

satisfies $\|g_s\|_2\leqq 2$. It is easily seen from the definition that the sets V_s are disjoint and hence $g_s\perp g_{s'}$ for $s\neq s'$. From (52), for $j\notin E_s$, $j\leqq J_s$

$$\langle f(jx),g_s\rangle=\frac{|jV|}{|V|}=1$$

and invoking (50), for

$$S_{J_s} f(x) = \sigma_{J_s}^{-1} \sum_{j \leqq J_s} \lambda_j f(jx) \quad \text{and} \quad \sigma_{J_s} = \sum_{j \leqq J_s} \lambda_j$$

we have

$$\langle S_{J_s} f, g_s \rangle \geqq \sigma_{J_s}^{-1} \sum_{\substack{j \leqq J_s \\ j \notin E_s}} \lambda_j \langle f(jx), g_s \rangle > \frac{1}{2}. \tag{53}$$

As in Lemma 6, the orthogonality of the g_s implies that the $\frac{1}{10}$-entropy-number of $\{S_{J_s} f \mid s \leqq r\}$ as subset of L^2 is at least proportional to r. Hence the entropy condition

$$\sup_{\|f\|_2 \leqq 1} N_f(\tfrac{1}{10}) < \infty$$

is violated, completing the proof.

We conclude this paper by applying the entropy test to the problem of A. Bellow [Be] on the averages

$$S_n f = \frac{1}{n} \sum_{j \leqq n} f_{a_j} \qquad f_a(x) = f(x + a)$$

where f is a function on T and the context is that of Proposition 3.

PROPOSITION 10. *Let $\{a_j\}$ be any sequence converging to 0 ($a_j \neq 0$ for each j). Then there exists a bounded measurable function f on* T *such that $(S_n f)$ is not a.s. converging (to f).*

In order to show that for some $\delta > 0$

$$\sup_{\|f\|_2 \leqq 1} N_f(\delta) = \infty, \tag{54}$$

the following simple lemma will be used.

LEMMA 11. *Let $\{a_j\}$ be a sequence of real numbers converging to 0. Then, given a positive integer r, there are integers $J_1 < J_2 < \cdots < J_r$ satisfying the following condition:*

Given a sequence $\bar{\alpha} = (\alpha_1, \ldots, \alpha_r)$ where $\alpha_s = 0$ or $\alpha_s = 1$, there is an integer $n = n(\bar{\alpha})$ such that for $s = 1, \ldots, r$

(55)
$$\left|1 - J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n}\right| < \frac{1}{10} \qquad \text{if } \alpha_s = 0$$
$$\left|1 - J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n}\right| > \frac{1}{2} \qquad \text{if } \alpha_s = 1.$$

(54) is then proved by considering the function

$$f = 2^{-r/2} \sum_{\bar{\alpha} \in \{0,1\}^r} e^{2\pi i n(\bar{\alpha})x}$$

for which

$$S_{J_s} f = 2^{-r/2} \sum_{\bar{\alpha} \in \{0,1\}^r} \beta_{s,\bar{\alpha}} e^{2\pi i n(\bar{\alpha})x}$$

letting

(56)
$$\beta_{s,\bar{\alpha}} = J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n(\bar{\alpha})}.$$

Thus, by (55),

(57)
$$|\beta_{s,\bar{\alpha}} - \beta_{t,\bar{\alpha}}| \geqq \tfrac{1}{2} - \tfrac{1}{10} = \tfrac{1}{3} \qquad \text{if } \alpha_s \neq \alpha_t.$$

Consequently, from (56), for $1 \leqq s \neq t \leqq r$, by (57)

$$\| S_{J_s} f - S_{J_t} f \|_2 = 2^{-r/2} \left(\sum_{\bar{\alpha}} |\beta_{s,\bar{\alpha}} - \beta_{t,\bar{\alpha}}|^2 \right)^{1/2} > \frac{1}{6}$$

and we may thus take $\delta = \frac{1}{6}$ in (54).

Proof of Lemma 11. For fixed $\bar{\alpha}$, integer $n = n(\bar{\alpha})$ will have the form

(58)
$$n = n_1 + n_2 + \cdots + n_r \qquad \text{where } n_s = \alpha_s m_s$$

and where the sequences $\{m_s\}$ and $\{J_s\}$ will be constructed simultaneously. Estimate

(59)
$$\left| J_s^{-1} \sum_{j \leq J_s} \exp(2\pi i a_j n) - J_s^{-1} \sum_{j \leq J_s} \exp(2\pi i a_j n_s) \right|$$
$$\leqq J_s^{-1} \sum_{j \leq J_s} \left| \exp\left[2\pi i a_j \left(\sum_{t<s} n_t \right) \right] - 1 \right| + J_s^{-1} \sum_{j \leq J_s} \left| \exp\left[2\pi i a_j \left(\sum_{t>s} n_t \right) \right] - 1 \right|.$$

Estimate the first term in (59) as

(60)
$$\frac{1}{50} + C \left(\sup_{j > J_s/100} |a_j| \right) \left(\sum_{t<s} m_t \right),$$

splitting the summation in $\Sigma_{j<J_s/100}$ and $\Sigma_{J_s/100\leqq j\leqq J_s}$ and using the inequality $|e^{2\pi i\lambda}-1|\leqq 2|\lambda|$.

Estimate the second term in (59) as

$$\sum_{t>s}\sup_{j\leqq J_s}|1-\exp(2\pi i a_j n_t)|<\frac{1}{50} \tag{61}$$

provided, since $n_t=\alpha_t m_t$,

$$\sup_{j\leqq J_{t-1}}|1-\exp(2\pi i a_j m_t)|<\frac{1}{50r}\qquad(1\leqq t\leqq r). \tag{62}$$

Assume now J_t, m_t obtained for $t<s$. Since $a_j\to 0$, J may be chosen such that (60) is at most $\frac{1}{40}$ provided $J_s>J$. We take integers m_s, J_s fulfilling the conditions $J_s>J_{s-1}+J$,

$$|1-\exp(2\pi i a_j m_s)|<\frac{1}{50r}\qquad\text{for } j\leqq J_{s-1}, \tag{63}$$

$$\Big|1-J_s^{-1}\sum_{j\leqq J_s}\exp(2\pi i a_j m_s)|>\frac{3}{4}. \tag{64}$$

Since $n_s=\alpha_s m_s$, (55) clearly follows from (59), (64) and the estimate $\frac{1}{50}+\frac{1}{40}$ on the right member of (59). It remains to show the existence of integers m_s, J_s satisfying (63), (64). Fix first a number T (depending on $a_j, j\leqq J_{s-1}$) such that the image of the interval $\mathbf{Z}\cap[-T,T]$ is ε-dense in the range of the map

$$\mathbf{Z}\to\mathbf{T}^{J_{s-1}}: z\mapsto(e^{2\pi i a_j z})_{j\leqq J_{s-1}}.$$

Here $\varepsilon=1/50r$. Thus to each $z\in\mathbf{Z}$ corresponds $t\in\mathbf{Z}$, $|t|\leqq T$ such that

$$|e^{2\pi i a_j z}-e^{2\pi i a_j t}|<\frac{1}{50r},\qquad j\leqq J_{s-1}. \tag{65}$$

Since $a_j\to 0$, J_s may be taken large enough to ensure

$$J_s^{-1}\sum_{j\leqq J_s}|a_j|<\frac{1}{100T}. \tag{66}$$

Next, since the $a_j\neq 0$,

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R\left\{\frac{1}{J_s}\sum_{j\leqq J_s}e^{2\pi i a_j x}\right\}dx=0,$$

implying the existence of some positive number y such that

$$\mathrm{Re}\left\{J_s^{-1}\sum_{j\le J_s} e^{2\pi i a_j y}\right\}<\frac{1}{10}. \tag{67}$$

Take $z=[y]$ (integer part) and t satisfying (65). Let $m_s=z-t$. Then, by (66), (67)

$$\begin{aligned}\mathrm{Re}\left\{J_s^{-1}\sum_{j\le J_s} e^{2\pi i a_j m_s}\right\} &\leqq \mathrm{Re}\left\{J_s^{-1}\sum_{j\le J_s} e^{2\pi i a_j y}\right\}+J_s^{-1}\sum_{j\le J_s}\left|1-e^{2\pi i a_j(y-z+t)}\right| \\ &\leqq \frac{1}{10}+2J_s^{-1}\sum_{j\le J_s}|a_j|(T+1) \\ &<\frac{1}{4}\end{aligned}$$

implying (64).

This completes the proof of Lemma 11 and hence Proposition 10.

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